4.10 NON-LINEAR ODEs

<u>CASE 1</u>: DEPENDENT VARIABLES ONLY APPEARS WITH DERIVATIVES

(IE. NO y, ONLY $y', y'', \ldots$)

LET $u = y' = \frac{dy}{dx}$

$$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{du}{dx} = u'$$

+ GET A DE IN x, u

eg. $y'y'' = 4x$

LET $u = y' = \frac{dy}{dx}$

$uu' = 4x$

$u\frac{du}{dx} = 4x$

$\int u\,du = \int 4x\,dx$

$\frac{1}{2}u^2 = 2x^2 + C$

$\frac{dy}{dx} = u = \pm\sqrt{4x^2 + C}$

$y = \pm\int\sqrt{4x^2 + C}\,dx$

IF $C = 0$

$y = \pm\int 2x\,dx$

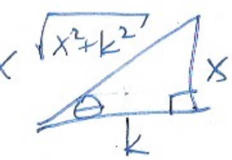
$y = \pm x^2 + K$

IF $C > 0$

LET $C = 4k^2$ $(k > 0)$

$y = \pm\int\sqrt{4x^2 + 4k^2}\,dx$

$= \pm 2\int\sqrt{x^2 + k^2}\,dx$



LET $x = k\tan\theta$

$\left(\tan\theta = \frac{x}{k}\right)$

$dx = k\sec^2\theta\,d\theta$

$$y = \pm 2 \int \sqrt{x^2+k^2}\, dx$$

$$= \pm 2 \int \sqrt{k^2\tan^2\theta + k^2}\; k\sec^2\theta\, d\theta$$

$$= \pm 2k^2 \int \sqrt{\tan^2\theta + 1}\; \sec^2\theta\, d\theta$$

$$= \pm 2k^2 \int \sec^3\theta\, d\theta$$

$$= \pm \cancel{2}k^2 \left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right) + K$$

$$\boxed{\begin{aligned} &\int \sec^3\theta\, d\theta \\ &= \tfrac{1}{2}\left(\text{DERIV OF } \sec\theta + \text{ANTI-DERIV OF } \sec\theta\right) \\ &= \tfrac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right) \end{aligned}}$$

$$= \pm k^2 \left(\frac{\sqrt{x^2+k^2}}{k}\frac{x}{k} + \ln\left| \frac{\sqrt{x^2+k^2}}{k} + \frac{x}{k} \right| \right) + K$$

$$= \pm \left(x\sqrt{x^2+k^2} + k^2 \ln \Big| \underbrace{\sqrt{x^2+k^2} + x}_{\sqrt{x^2+k^2}+x>0} \Big| \right) + K$$

$$= \pm \left(x\sqrt{x^2+k^2} + k^2 \ln\left(\sqrt{x^2+k^2} + x\right)\right) + K$$

or $$\pm \left(x\sqrt{x^2+C} + C\ln\left(\sqrt{x^2+C} + x\right)\right) + K$$

$$C > 0$$

$$\ln\left| \frac{\sqrt{x^2+k^2}}{k} + \frac{x}{k} \right|$$

$$= \ln\left| \frac{\sqrt{x^2+k^2} + x}{k} \right|$$

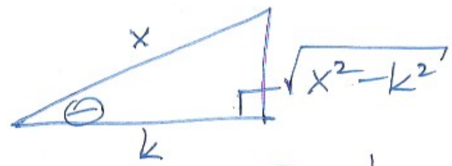
$$= \ln\left|\sqrt{x^2+k^2} + x\right| \underbrace{- \ln|k|}_{\text{ARBITRARY CONSTANT}}$$

IF $C < 0$

LET $C = -4k^2 \quad (k > 0)$

$$y = \pm \int \sqrt{4x^2 - 4k^2}\, dx$$

$$= \pm 2 \int \sqrt{x^2 - k^2}\, dx$$



$x = k\sec\theta \rightarrow dx = k\sec\theta\tan\theta\, d\theta$

$\sec\theta = \frac{x}{k}$

$$= \pm 2 \int \sqrt{k^2\sec^2\theta - k^2} \cdot k\sec\theta\tan\theta\, d\theta$$

$$= \pm 2k^2 \int \tan^2\theta \sec\theta\, d\theta$$

$$= \pm 2k^2 \int (\sec^2\theta - 1)\sec\theta\, d\theta$$

$$= \pm 2k^2 \left(\int \sec^3\theta\, d\theta - \int \sec\theta\, d\theta \right)$$

$$= \pm 2k^2 \left(\tfrac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|) - \ln|\sec\theta + \tan\theta| \right) + K$$

$$= \pm \cancel{2}k^2 \left(\cancel{\tfrac{1}{2}}\sec\theta\tan\theta - \cancel{\tfrac{1}{2}}\ln|\sec\theta + \tan\theta| \right) + K$$

$$= \pm k^2 \left(\frac{x}{k}\frac{\sqrt{x^2-k^2}}{k} - \ln\left|\frac{x}{k} + \frac{\sqrt{x^2-k^2}}{k}\right| \right) + K$$

$$= \pm \left(x\sqrt{x^2-k^2} - k^2\ln|x + \sqrt{x^2-k^2}| \right) + K$$

$C = -k^2$

$C < 0$

$$= \pm \left(x\sqrt{x^2+C} + C\ln|x + \sqrt{x^2+C}| \right) + K$$

$$y = \pm\left(x\sqrt{x^2+C} + C\ln\left|\sqrt{x^2+C}+x\right|\right)+K \quad \text{IF } C\neq 0$$

$y =$ IF $C=0$

$\pm x^2+K$

NOTE: IF $C=0$

$$y = \pm\left(x\sqrt{x^2}+0\right)+K$$

$$y = \pm x^2+K$$

SO $$y = \pm\left(x\sqrt{x^2+C} + C\ln\left|\sqrt{x^2+C}+x\right|\right)+K, \quad C K\in\mathbb{R}$$

CASE 2: NO INDEPENDENT VARIABLE (EXCEPT AS dx)

LET $u = y' = \frac{dy}{dx}$ AND MAKE y THE INDEPENDENT VARIABLE

$$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = u'u$$

$$u' = \frac{du}{dy}$$

eg. $y^2 y'' = y'$

$y^2 u' \cancel{u} = \cancel{u}$ $\longrightarrow$ IS $u=0$ A SOL'N? $\frac{dy}{dx} = 0$

IS $y = C$ A SOL'N? $C^2 \cdot 0 = 0$ ✓

$y^2 \frac{du}{dy} = 1$

$\int du = \int \frac{1}{y^2} dy$

$\frac{dy}{dx} = u = -\frac{1}{y} + C \longrightarrow$ WHAT IF $C=0$?

$\frac{dy}{dx} = -\frac{1}{y}$

$\int y\,dy = \int dx$

$-\frac{1}{2}y^2 = x + K$

$y^2 = K - 2x$

$y = \pm\sqrt{K-2x}$ IS ALSO A SINGULAR SOL'N

$$\frac{dy}{dx} = -\frac{1}{y} + C = \frac{Cy-1}{y}$$

$$\begin{array}{r|l} & \frac{1}{C} \\ \hline Cy-1 & y \\ & -y + \frac{1}{C} \\ \hline & \frac{1}{C} \end{array}$$

$$\int \frac{y}{Cy-1} dy = \int dx$$

$$\int \left(\frac{1}{C} + \frac{\frac{1}{C}}{Cy-1}\right) dy = x + K$$

$$\frac{1}{C} y + \frac{1}{C^2}\ln|Cy-1| = x + K$$

$$Cy + \ln|Cy-1| = C^2 x + K$$

$y = C$ IS A SINGULAR SOL'N